Tutorial sheet 07

Topics covered

- Introduction to Linear Algebra
- Eigenvalues and Eigenvectors

Using Matlab for matrix operations

Action	Matlab command window
Create a row vector	**x = [5 6 7]**
Create a column vector	**x = [5 6 7]'** **x = [5; 6; 7]**
Create a matrix	**A = [5 6 7; 8 9 10; 11 12 13]**
Access/change matrix element value	**A(1, 2) = 0**
Add/subtract matrices	**C = A + B** **C = A - B**
Multiply matrices	**C = A * B**
Determinant	**det(A)**
Inverse matrix	**inv(A)**
Matrix eigenvalues	**eig(A)**
Matrix eigenvectors	**[V, D] = eig(A)** (V - matrix of eigenvectors, D - matrix with eigenvalues on the diagonal)

Using Matlab to calculate eigenvalues and eigenvectors

To use Matlab to find eigenvalues and eigenvectors of a matrix:
(1) clear environment if needed: **clear**
(2) enter matrix, e.g.:
A = [-10 5 0; 5 -10 5; 0 10 -10];
(3) calculate eigenvalues and eigenvectors:
[V,D]=eig(A)
(4) if needed, extract eigenvalues which are diagonal elements of matrix D:

lambda1 = D(1,1); lambda2 = D(2,2); lambda3 = D(3,3);

(5) if needed, extract eigenvectors which are columns of matrix V:

v1=V(:,1); v2=V(:,2); v3=V(:,3)

Problems: basic level

7.01 Calculate volume of a parallelogram formed by three vectors:

(a) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

(b) $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$

7.02 If possible find *AB* and *BA*:

$$A = \begin{bmatrix} -2 & 1 & 7 \\ 3 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 4 & -1 & 5 \end{bmatrix}$$

7.03 For given matrices find det(A), det(B), $2A+3B$, A^2, B^2, AB, BA, A^T, AB^T and B^{-1}. Solve the problem analytically and using **Matlab**

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 1 & -1 \\ 0 & 5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 1 \\ 5 & -2 & 0 \\ 1 & -4 & 1 \end{bmatrix}$$

7.04 Find eigenvalues and eigenvectors of these matrices (analytically)

(a) $A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 5 & 2 \\ -8 & -3 \end{bmatrix}$ (c) $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ (d) $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

(e) $A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$ (f) $A = \begin{bmatrix} 3 & -2 \\ -4 & -1 \end{bmatrix}$

7.05 Find eigenvalues and eigenvectors of these matrices using Matlab

(a) $A = \begin{bmatrix} 3 & 2 & -2 \\ -3 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & 0 & 3 \end{bmatrix}$ (c) $A = \begin{bmatrix} 15 & 7 & -7 \\ -1 & 1 & 1 \\ 13 & 7 & -5 \end{bmatrix}$

Problems: standard level

7.06 For matrix

$$A = \frac{1}{\sqrt{7}} \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$$

find A^2, A^3 and A^{10}

7.07 Two linear transformations are described by matrices *A* and *B*. If we apply first transformation *A* and then *B*, what is the matrix of the resulting transformation *BA*? What is the result if we apply them in the reverse order? Solve the problem analytically and using **Matlab**

(a) $A=\begin{bmatrix}5 & -1 & 4\\ 3 & -2 & -1\\ 0 & 4 & 0\end{bmatrix}$, $B=\begin{bmatrix}4 & 0 & 1\\ 2 & -2 & 0\\ 1 & 4 & 1\end{bmatrix}$

(b) $A=\begin{bmatrix}0 & 1 & 2\\ 3 & -4 & -5\\ 2 & 4 & 3\end{bmatrix}$, $B=\begin{bmatrix}1 & 0 & 1\\ -4 & -5 & 2\\ -2 & -13 & 1\end{bmatrix}$

7.08 (intmath.com) In studying the motion of electrons, one of the Pauli spin matrices is

$$s=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$$

Here *i* is imaginary unit (sometimes denotes as *j*). Show that $s^2 = I$ (I is a 2x2 identity matrix)

7.09 Find eigenvalues and eigenvectors of these matrices (analytically and using Matlab)

(a) $A=\begin{bmatrix}0.2 & -0.1\\ 0.4 & 0.6\end{bmatrix}$ (b) $A=\begin{bmatrix}3 & 5\\ 0 & 3\end{bmatrix}$ (c) $A=\begin{bmatrix}3.16 & -5.2\\ 0.45 & -2.5\end{bmatrix}$

(d) $A=\begin{bmatrix}2 & -1\\ 5 & -2\end{bmatrix}$ (e) $A=\begin{bmatrix}3 & -2\\ 4 & -1\end{bmatrix}$ (f) $A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$

7.10 Find eigenvalues and eigenvectors of the following matrices (using Matlab):

(a) $A=\begin{bmatrix}-2 & 0 & 1\\ 1 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$ (b) $A=\begin{bmatrix}5 & -4 & 4\\ 12 & -11 & 12\\ 4 & -4 & 5\end{bmatrix}$

7.11 For the matrix

$$A=\begin{bmatrix}1 & -1 & 0\\ 2 & 4 & 0\\ -2 & -1 & 1\end{bmatrix}$$

and its inverse, A^{-1}, determine the eigenvalues and a set of corresponding eigenvectors. Comment on the result.

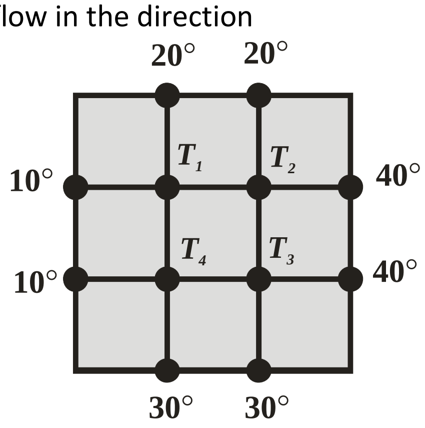
Problems: advanced level

7.12 Consider a cross-section of a metal beam with negligible heat flow in the direction perpendicular to the plane. Let T_1, T_2, T_3 and T_4 represent temperatures at the four interior nodes in the mesh in the figure.

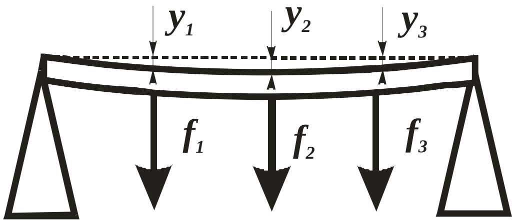
The temperature at a node is approximately equal to the average of the four nearest nodes, for example

$$T_1 = (10 + 20 + T_2 + T_4)/4$$

Write a system of equations for temperatures T_1, T_2, T_3 and T_4 and solve it using **Matlab**.



7.13 A horizontal elastic beam is supported at each end and is subjected to forces at points 1, 2 and 3.



If y is the deflection at each of three points, it can be described (using Hooke's law) as

$$\mathbf{y} = D\mathbf{f}\text{, where } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \text{ and } \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Matrix D is called flexibility matrix.

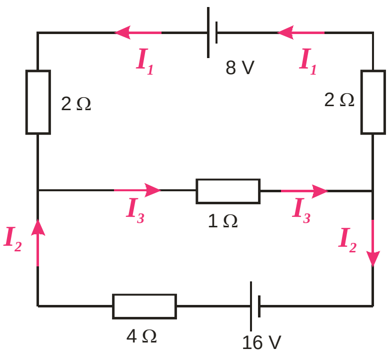
For given flexibility matrices D find the inverse matrix $K = D^{-1}$ which is called stiffness matrix, and then calculate the forces if deflections are known (analytically and using **Matlab**):

(a) $D = \begin{bmatrix} 0.005 & 0.002 & 0.002 \\ 0.002 & 0.004 & 0.002 \\ 0.001 & 0.002 & 0.005 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0.06 \\ 0.09 \\ 0.05 \end{bmatrix}$ (m)

(b) $D = \begin{bmatrix} 0.001 & 0.003 & 0.003 \\ 0.001 & 0.004 & 0.003 \\ 0.001 & 0.003 & 0.004 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0.08 \\ 0.09 \\ 0.05 \end{bmatrix}$ (m)

7.14 The currents in the electrical network satisfy a set of linear equations

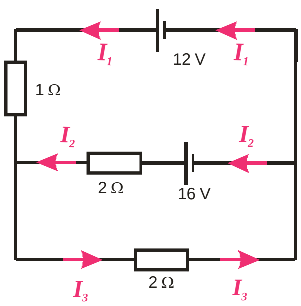
$$I_1 + I_2 - I_3 = 0$$
$$4I_1 + I_3 = 8$$
$$4I_2 + I_3 = 16$$



Write down the system in matrix form and use inverse matrix to find electric currents I_1, I_2 and I_3. Solve the problem analytically and using **Matlab**

7.15 The currents in the electrical network satisfy a set of linear equations

$$I_1 + I_2 - I_3 = 0$$
$$I_1 + 2I_3 = 12$$
$$I_1 - 2I_2 = -4$$



Write down the system in matrix form and use inverse matrix to find electric currents I_1, I_2 and I_3. Solve the problem analytically and using **Matlab**

7.16 Capital letter N is determined by 8 points or vertices. Each of them can be represented by an appropriate position vector, and the resulting matrix of the position vectors is:

$$\begin{array}{r} \\ \\ x-coordinate \\ y-coordinate \end{array} \begin{array}{c} Vertex: \\ \begin{array}{cccccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} \end{array} \\ \begin{bmatrix} 0 & 0.5 & 0.5 & 6 & 6 & 5.5 & 5.5 & 0 \\ 0 & 0 & 6.42 & 0 & 8 & 8 & 1.58 & 8 \end{bmatrix} \end{array}$$

Shear transformation is defined by a matrix:

$$A=\begin{bmatrix}1 & 0.25\\ 0 & 1\end{bmatrix}$$

Calculate the points of letter *N* after the transformation *A* is applied. Sketch letter N before and after transformation (manually or using Matlab.)

7.17 If in the previous example we also want to scale the letter by a factor of 0.75 using the transformation matrix

$$B=\begin{bmatrix}0.75 & 0\\ 0 & 1\end{bmatrix}$$

what would be the resulting transformation? Plot letter N after the transformation is applied (manually or using Matlab).

7.18 (intmath.com) Evaluate the following matrix multiplication which is used in directing the motion of a robotic mechanism, and interpret the result (i.e. the meaning of the transformation):

$$\begin{pmatrix}\cos 60^\circ & -\sin 60^\circ & 0\\ \sin 60^\circ & \cos 60^\circ & 0\\ 0 & 0 & 1\end{pmatrix}\times\begin{pmatrix}2\\4\\0\end{pmatrix}$$

7.19 Find eigenvalues and eigenvectors of this matrix (analytically and using Matlab)

$$A=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$$

Answers to Tutorial exercises

7.01 (a) -2; (b) 1

7.02 $$BA=\begin{bmatrix}4 & -1 & 5\end{bmatrix}\begin{bmatrix}-2 & 1 & 7\\ 3 & -1 & 0\\ 0 & 2 & -1\end{bmatrix}=\begin{bmatrix}-11 & 15 & 23\end{bmatrix}$$

AB does not exist.

7.03

det(A) = 70

det(B) = -24

$$2A+3B=\begin{bmatrix}13 & -2 & 11\\ 21 & -4 & -2\\ 3 & -2 & 3\end{bmatrix},\quad A^2=\begin{bmatrix}1 & 17 & 9\\ 9 & -7 & 11\\ 15 & 5 & -5\end{bmatrix},\ B^2=\begin{bmatrix}10 & -4 & 4\\ 5 & 4 & 5\\ -16 & 4 & 2\end{bmatrix}$$

$$AB=\begin{bmatrix}5 & -14 & 6\\ 13 & 2 & 2\\ 25 & -10 & 0\end{bmatrix},\ BA=\begin{bmatrix}6 & 2 & 12\\ 4 & -7 & 22\\ -10 & 0 & 8\end{bmatrix},$$

$$A^T=\begin{bmatrix}2 & 3 & 0\\ -1 & 1 & 5\\ 4 & -1 & 0\end{bmatrix},\quad AB^T=\begin{bmatrix}10 & 12 & 10\\ 8 & 13 & -2\\ 0 & -10 & -20\end{bmatrix},\ B^{-1}=\begin{bmatrix}\frac{1}{12} & \frac{1}{6} & -\frac{1}{12}\\ \frac{5}{24} & -\frac{1}{12} & -\frac{5}{24}\\ \frac{3}{4} & -\frac{1}{2} & \frac{1}{4}\end{bmatrix}$$

7.04 (a) $\lambda_1 = 0$, $\lambda_2 = 4$, $\mathbf{x}_1 = \alpha\begin{bmatrix}2\\1\end{bmatrix}$, $\mathbf{x}_2 = \beta\begin{bmatrix}-2\\1\end{bmatrix}$

(b) $\lambda_{1,2} = 1$, $\mathbf{x}_{1,2} = \alpha\begin{bmatrix}1\\-2\end{bmatrix}$

(c) $\lambda_{1,2} = 3$, $\mathbf{x}_{1,2} = \alpha\begin{bmatrix}1\\1\end{bmatrix}$

(d) $\lambda_1 = 1$, $\lambda_2 = 6$, $\mathbf{x}_1 = \alpha\begin{bmatrix}1\\-1\end{bmatrix}$, $\mathbf{x}_2 = \beta\begin{bmatrix}4\\1\end{bmatrix}$

(e) $\lambda_1 = \frac{1+\sqrt{21}}{2} = 2.79$, $\lambda_2 = \frac{1-\sqrt{21}}{2} = -1.79$, $\mathbf{x}_1 = \alpha\begin{bmatrix}2\\\sqrt{21}-3\end{bmatrix} = \alpha\begin{bmatrix}2\\1.58\end{bmatrix}$

$\mathbf{x}_2 = \beta\begin{bmatrix}2\\-\sqrt{21}-3\end{bmatrix} = \beta\begin{bmatrix}2\\-7.58\end{bmatrix}$

(f) $\lambda_1 = -2.46$, $\lambda_2 = 4.46$, $\mathbf{x}_1 = \alpha\begin{bmatrix}1\\2.73\end{bmatrix}$, $\mathbf{x}_2 = \beta\begin{bmatrix}1\\-0.73\end{bmatrix}$

7.05 (a) $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 2$, $\mathbf{x}_1 = \alpha\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\mathbf{x}_2 = \beta\begin{bmatrix}1\\-1\\1\end{bmatrix}$, $\mathbf{x}_3 = \gamma\begin{bmatrix}0\\1\\1\end{bmatrix}$

(b) $\lambda_{1,2} = 1$, $\lambda_3 = 3$, $\mathbf{x}_2 = \alpha\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{x}_2 = \beta\begin{bmatrix}0\\1\\1\end{bmatrix}$

(c) $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 8$, $\mathbf{x}_1 = \alpha\begin{bmatrix}1\\-1\\1\end{bmatrix}$, $\mathbf{x}_2 = \beta\begin{bmatrix}0\\1\\1\end{bmatrix}$, $\mathbf{x}_3 = \gamma\begin{bmatrix}1\\0\\1\end{bmatrix}$

7.06 $A^2 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, $A^3 = \frac{1}{\sqrt{7}}\begin{bmatrix}2 & 3\\1 & -2\end{bmatrix}$, $A^4 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ and so on: $A^{10} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

7.07 (a) $BA = \begin{bmatrix}20 & 0 & 16\\4 & 2 & 10\\17 & -5 & 0\end{bmatrix}$, $AB = \begin{bmatrix}22 & 18 & 9\\7 & 0 & 2\\8 & -8 & 0\end{bmatrix}$

(b) $BA = \begin{bmatrix}2 & 5 & 5\\-11 & 24 & 23\\-37 & 54 & 64\end{bmatrix}$, $AB = \begin{bmatrix}-8 & -31 & 4\\29 & 85 & -10\\-20 & -59 & 13\end{bmatrix}$

7.09 (a) $\lambda = 0.4$, $\mathbf{x}_1 = \alpha\begin{bmatrix}1\\-2\end{bmatrix}$.

(b) $\lambda = 3$, $\mathbf{x}_1 = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(c) $\lambda_1 = 2.71$, $\lambda_2 = -2.05$, $\mathbf{x}_1 = \alpha \begin{bmatrix} 1 \\ 0.0865 \end{bmatrix}$, $\mathbf{x}_2 = \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(d) $\lambda_{1,2} = \pm i$, $\mathbf{x}_1 = \alpha \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$, $\mathbf{x}_2 = \beta \begin{bmatrix} 1 \\ 2+i \end{bmatrix}$

(e) $\lambda_1 = 1+2i$, $\lambda_2 = 1-2i$, $\mathbf{x}_1 = \alpha \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$, $\mathbf{x}_2 = \beta \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

(f) $\lambda_1 = i$, $\lambda_2 = -i$, $\mathbf{x}_1 = \alpha \begin{bmatrix} -i \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \beta \begin{bmatrix} i \\ 1 \end{bmatrix}$

7.10 (a) $\lambda_1 = 1,\ \lambda_2 = \lambda_3 = -2,\ \mathbf{x}_1 = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{x}_2 = \mathbf{x}_3 = \beta \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

(b) $\lambda_1 = -3,\ \lambda_2 = \lambda_3 = 1,\ \mathbf{x}_1 = \alpha \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix},\ \mathbf{x}_2 = \beta \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},\ \mathbf{x}_3 = \gamma \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

7.11 $\lambda_1 = 3,\ \lambda_2 = 2,\ \lambda_3 = 1,\ \mathbf{x}_1 = \alpha \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix},\ \mathbf{x}_2 = \beta \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{x}_3 = \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Eigenvalues of A^{-1} are $\lambda_1 = \frac{1}{3},\ \lambda_2 = \frac{1}{2},\ \lambda_3 = 1$ with the same eigenvectors.

7.12 $A = \begin{bmatrix} 4 & -1 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ -1 & 0 & -1 & 4 \end{bmatrix},\ b = \begin{bmatrix} 30 \\ 60 \\ 70 \\ 40 \end{bmatrix},\ A^{-1} = \frac{1}{24} \begin{bmatrix} 7 & 2 & 1 & 2 \\ 2 & 7 & 2 & 1 \\ 1 & 2 & 7 & 2 \\ 2 & 1 & 2 & 7 \end{bmatrix},$

$T_1 = 20$, $T_2 = 27.5$, $T_3 = 30$, $T_4 = 22.5$

7.13 (a) $D^{-1} = \begin{bmatrix} 250 & -93.75 & -62.5 \\ -125 & 359.375 & -93.75 \\ 0 & -125 & 250 \end{bmatrix},\ \mathbf{f} = \begin{bmatrix} 3.4375 \\ 20.15625 \\ 1.25 \end{bmatrix}$ (N)

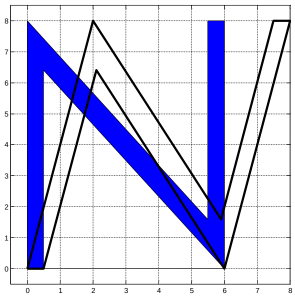
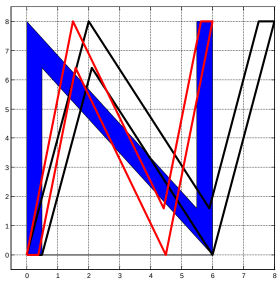
(b) $D^{-1} = \begin{bmatrix} 7000 & -3000 & -3000 \\ -1000 & 1000 & 0 \\ -1000 & 0 & 1000 \end{bmatrix},\ \mathbf{f} = \begin{bmatrix} 140 \\ 10 \\ -30 \end{bmatrix}$ (N)

7.14 $I_1 = 1,\ I_2 = 3,\ I_3 = 4$ (A)

7.15 $I_1 = 2,\ I_2 = 3,\ I_3 = 5$ (A)

7.16 and 7.17:

$\begin{bmatrix} 0 & 0.5 & 2.1050 & 6 & 8 & 7.5 & 5.895 & 2 \\ 0 & 0 & 6.42 & 0 & 8 & 8 & 1.58 & 8 \end{bmatrix},\ BA = \begin{bmatrix} 0.75 & 0.1875 \\ 0 & 1 \end{bmatrix}$

7.18 $\begin{pmatrix} -2.464 \\ 3.732 \\ 0 \end{pmatrix}$. The transformation resulted in robotic arm rotating 60° in (x,y) plane.

7.19 $\lambda_{1,2} = 3$, $\mathbf{x}_1 = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$